Analysis of the Hypertriton in Terms of Hard-Core Potentials*

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The variation method was used to set upper bounds on the strength of the average Λ-nucleon potential in the hypertriton ($_{\Lambda}H^3$) required to reproduce the observed binding energy of that system for assumed hard-core radii of 0.2 F, 0.4 F, and 0.6 F. Only two-body Λ-nucleon potentials were considered. Although the well-depth parameter of the required potential increases as the assumed hard-core radius is increased, it seems unlikely that even the largest of the hard-core radii considered here would imply a bound state for the hyperdeuteron ($_{\Lambda}H^2$). The possibility that the scattering length and the effective range of the required potential may be insensitive to the value of the hard-core radius is discussed.

I. INTRODUCTION

RECENT phenomenological nucleon-nucleon interaction potentials, which have been deduced from analyses of nucleon-nucleon scattering data and from the observed properties of the deuteron, contain a hard core of radius about 0.5 F.[1] The scattering data seem to require, for their explanation, the presence of a short-range repulsion in the interaction in most, if not all, states; this repulsion is usually represented by a hard core.[1] The presence of a hard core in the nucleon-nucleon potential suggests that a hard core may also be a characteristic of the Λ-nucleon interaction potential.[2]

In the absence of extensive Λ-nucleon scattering data and on account of the apparent nonexistence of a bound state of the Λ-nucleon system (hyperdeuteron), attempts to deduce the features of the Λ-nucleon interaction have been directed toward analyses of the binding-energy data of the established hypernuclei with $A\geqslant 3$.[3] These analyses have so far been aimed at establishing gross features of the Λ-nucleon interaction; potentials which have been used are spin-dependent central potentials which have been considered to contain the effect of a possible tensor component.[2–4]

Since analyses of hypernuclear binding energy data have been made in terms of effective central potentials, it is important to ask what effect the possible presence of a hard core can be expected to have in such analyses. An indication of the importance of a hard core in effective central potentials in reproducing observed binding energies can be obtained from studies of the very light nuclei in terms of such potentials. A consistent reproduction of the binding energies of the two-, three-, and four-nucleon systems has been obtained with effective central potentials having a hard core.[5–7] On the other hand, when effective central potentials without hard cores, which are consistent with the two-nucleon binding energy and low-energy scattering data, are used in variation calculations, they lead to binding energies for the triton and the alpha particle in excess of the empirical values.[8–10] It would,

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[1] See, for example, T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962), and the review by M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nucl. Sci. 11, 95 (1961).

[2] For example, the assumption of the existence of a universal pion-baryon interaction leads to a pion-exchange contribution to the Λ-nucleon potential which is a linear combination of nucleon-nucleon potentials [see, for example, D. B. Lichtenberg and M. Ross, Phys. Rev. 107, 1714 (1957)]. In this case, the existence of a hard core and a tensor component in the nucleon-nucleon interaction implies that these are also characteristics of the Λ-nucleon interaction.

[3] See, for example, R. H. Dalitz and B. W. Downs, Phys. Rev. 111, 967 (1958), R. H. Dalitz, *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961* (Heywood and Company, Ltd., London, 1961), p. 103, and A. R. Bodmer and S. Sampanthar, Nucl. Phys. 31, 251 (1962).

[4] Although there may well be a tensor component in the Λ-nucleon interaction (see footnote 2), uncertainties in analyses of hypernuclear binding energy data (see the references in footnote 3) would seem to make attempts to separate the effects of the central and tensor parts of the potential appear unpromising at this time.

[5] T. Ohmura (Kikuta), M. Morita, and M. Yamada, Progr. Theoret. Phys. (Kyoto) 15, 222 (1956); 17, 326 (1957).

[6] T. Ohmura, Progr. Theoret. Phys. (Kyoto) 22, 34 (1959). This paper corrected a systematic error in the papers of reference 5. On account of this error, the expectation values of the kinetic energy of the three-nucleon system given in reference 5 were about 2% too large ($\Delta T\approx 1$ MeV for $T\approx 50$ MeV).

[7] L. Cohen and J. B. Willis, *Nuclear Forces and the Few-Nucleon Problem*, edited by T. C. Griffith and E. A. Power (Pergamon Press, New York, 1960), p. 399, and H. C. Mang, W. Wild, and F. Beck, *ibid.*, p. 403.

[8] See, for example, J. Irving, Phil. Mag. 42, 338 (1951).

[9] The success of central hard-core potentials in reproducing the binding-energy data of the lightest nuclei does not, of course,

therefore, seem that analyses of the hypernuclear binding-energy data should be made in terms of hard-core potentials to complement the studies which have been made in terms of potentials without hard cores.[3]

In order to include the effect of a hard core in the Λ-nucleon interaction, Lichtenberg[11] adapted the results of the triton variation calculations of Ohmura (Kikuta), Morita, and Yamada[5] to the hypertriton for the case in which the range of the Λ-nucleon interaction is approximately the same as that of the nucleon-nucleon interaction.[6] Dietrich, Folk, and Mang[12] have recently used the independent-pair approximation of Gomes, Walecka, and Weisskopf[13] to deduce the parameters of Λ-nucleon potentials with an attractive square well and a hard-core radius of 0.2 F.

It is the purpose of this paper to report the results of variation calculations of the strength of the effective Λ-nucleon interaction in the hypertriton $_{\Lambda}H^3$ for several values of the hard-core radius. The attractive well was taken to have an exponential shape and a range corresponding to the simplest pion-exchange mechanism (two-pion exchange) which can give rise to a charge-independent Λ-nucleon interaction. This choice of range implies a nonsymmetric structure for the hypertriton[14]; whereas, in the situation investigated by Lichtenberg,[11] the structure was taken to be symmetric, which is probably not realistic.

The variation calculation is described in Sec. II, and the parameters of the potentials are discussed in Sec. III. The results of the calculations are given in Sec. IV and discussed in Sec. V, where a comparison is made with the work of Lichtenberg[11] and of Dietrich *et al.*[12] The implications of the results for the possible binding of the hyperdeuteron are also discussed.

imply that such potentials provide an adequate representation of the actual nucleon-nucleon interaction for the description of manifestations of that interaction other than these binding energies (see the references in footnote 1). In this connection, it should be noted that a detailed calculation of the binding energy of the triton by J. M. Blatt, G. H. Derrick, and J. N. Lyness, Phys. Rev. Letters **8**, 322 (1962), in terms of nucleon-nucleon potentials which reproduce a wide variety of two-body data, failed to reproduce the experimental value.

[10] It might be noted that the effect of explicit consideration of the tensor potential in calculations of the binding energies of the very light nuclei is qualitatively the same as the effect of the introduction of a hard core. The binding energy of the triton has been reproduced by R. L. Pease and H. Feshbach, Phys. Rev. **88**, 945 (1952) and the binding energy of the alpha particle has been approximately reproduced by J. Irving, Proc. Phys. Soc. (London) **A66**, 17 (1953) with nucleon-nucleon potentials which have a tensor component but not a hard core. Although the presence of a tensor component in the nucleon-nucleon interaction is well established (see, for example, the references in footnote 1), the same cannot be said of the Λ-nucleon interaction at this time (see footnote 4).

[11] D. B. Lichtenberg, Nuovo Cimento **8**, 463 (1958).

[12] K. Dietrich, R. Folk, and H. J. Mang, *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961* (Heywood and Company, Ltd., London, 1960), p. 165.

[13] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. (N. Y.) **3**, 241 (1958).

[14] See, for example, R. H. Dalitz and B. W. Downs, Phys. Rev. **110**, 958 (1958).

II. FORMULATION OF THE VARIATION PROBLEM

The nucleon-nucleon and Λ-nucleon interactions in the hypertriton were taken to be two-body central potentials with exponential attractive wells and the same hard-core radius D.[15]

Nucleon-nucleon:

$$\begin{aligned} V(r) &= \infty, & r<D \\ &= -V_0 \exp[-\eta(r-D)], & r>D, \end{aligned} \tag{2.1a}$$

Λ-nucleon:

$$\begin{aligned} U(r) &= \infty, & r<D \\ &= -U_0 \exp[-\lambda(r-D)], & r>D. \end{aligned} \tag{2.1b}$$

The nucleon-nucleon potential (2.1a), which is effective in the hypertriton, is that for the triplet spin state. It has been assumed in (2.1b) that the Λ-neutron and Λ-proton potentials are the same, in accordance with the charge independence of strong interactions; then $U(r)$ is the average Λ-nucleon potential effective in the hypertriton.[16]

The variation method was used to obtain an upper bound on the depth U_0 of the average Λ-nucleon potential required to reproduce the observed binding energy B_Λ of the Λ particle in the hypertriton. The appropriate variation inequality is

$$U_0 \leqslant [T+V+(B_d+B_\Lambda)N]/2P, \tag{2.2}$$

where N is the normalization integral, and T, V, and P are the expectation values of the total kinetic energy, the nucleon-nucleon potential (2.1a), and the average Λ-nucleon potential (2.1b) divided by $-U_0$, respectively; B_d is the binding energy of the deuteron.

It is convenient to formulate the variation problem (2.2) in the triangular coordinate system, in which the radial variables (r_1,r_2,r_3) are the three interparticle separations. In this coordinate system, it is possible to construct a trial wave function which takes into account the correlations between pairs of particles which is necessary if the variation method is to lead to reliable results.[14] The trial wave function, appropriate to the hard-core potentials (2.1), was taken to be

$$\psi = f(r_1)f(r_2)g(r_3), \tag{2.3}$$

with

$$\begin{aligned} f(r) &= 0, & r<D \\ &= \exp[-\alpha(r-D)]-\exp[-\beta(r-D)], & r>D, \end{aligned} \tag{2.4a}$$

$$\begin{aligned} g(r) &= 0, & r<D \\ &= \exp[-\gamma(r-D)]-\exp[-\delta(r-D)], & r>D, \end{aligned} \tag{2.4b}$$

[15] In the absence of contrary evidence, this assumption about the equality of the hard-core radii is made for computational convenience.

[16] The relevant average is U_{triplet} if the triplet interaction is the more attractive or $(3U_{\mathrm{singlet}}+U_{\mathrm{triplet}})/4$ if the singlet interaction is the more attractive. The average Λ-nucleon potential and the nucleon-nucleon potential are to be considered to be effective central potentials which include the effect of the appropriate tensor interactions.

describing the ground state in which each pair of particles is in a relative S state.[14] The separations r_1 and r_2 are those of the two Λ-nucleon pairs, and r_3 is that of the two nucleons. That the dependence of the wave function (2.3) on r_1 and r_2 is the same is consistent with the requirements of the generalized Pauli principle. The variation parameters (α,β) and (γ,δ) of the Λ-nucleon and nucleon-nucleon parts of the trial function are allowed to be different in order to take account of an asymmetric structure for the hypertriton.[14]

In the triangular coordinate system, the expectation value of the kinetic energy for a bound system can be expressed in two equivalent forms when the wave function depends only upon the interparticle separations, as it does in (2.3). These are[17]

$$T=\int\psi\bigg\{-\frac{\hbar^2(M_\Lambda+M)}{2MM_\Lambda}\left(\frac{\partial^2\psi}{\partial r_1^2}+\frac{2}{r_1}\frac{\partial\psi}{\partial r_1}+\frac{\partial^2\psi}{\partial r_2^2}+\frac{2}{r_2}\frac{\partial\psi}{\partial r_2}\right)-\frac{\hbar^2}{M}\left(\frac{\partial^2\psi}{\partial r_3^2}+\frac{2}{r_3}\frac{\partial\psi}{\partial r_3}\right)-\frac{\hbar^2}{2M_\Lambda}\frac{r_1^2+r_2^2-r_3^2}{r_1r_2}\frac{\partial^2\psi}{\partial r_1\partial r_2}$$
$$-\frac{\hbar^2}{2M}\left[\frac{r_2^2+r_3^2-r_1^2}{r_2r_3}\frac{\partial^2\psi}{\partial r_2\partial r_3}+\frac{r_3^2+r_1^2-r_2^2}{r_3r_1}\frac{\partial^2\psi}{\partial r_3\partial r_1}\right]\bigg\}r_1r_2r_3dr_1dr_2dr_3 \quad (2.5\text{a})$$

and

$$T=\int\bigg\{\frac{\hbar^2(M_\Lambda+M)}{2MM_\Lambda}\left[\left(\frac{\partial\psi}{\partial r_1}\right)^2+\left(\frac{\partial\psi}{\partial r_2}\right)^2\right]+\frac{\hbar^2}{M}\left(\frac{\partial\psi}{\partial r_3}\right)^2+\frac{\hbar^2}{2M_\Lambda}\frac{r_1^2+r_2^2-r_3^2}{r_1r_2}\frac{\partial\psi}{\partial r_1}\frac{\partial\psi}{\partial r_2}$$
$$+\frac{\hbar^2}{2M}\left[\frac{r_2^2+r_3^2-r_1^2}{r_2r_3}\frac{\partial\psi}{\partial r_2}\frac{\partial\psi}{\partial r_3}+\frac{r_3^2+r_1^2-r_2^2}{r_3r_1}\frac{\partial\psi}{\partial r_3}\frac{\partial\psi}{\partial r_1}\right]\bigg\}r_1r_2r_3dr_1dr_2dr_3. \quad (2.5\text{b})$$

For any product wave function of the form (2.3) [including the case in which $f(r_1)$ is different from $f(r_2)$] an expression for the expectation value of the kinetic energy, which is simpler than either (2.5a) or (2.5b), can be obtained by taking one-half the sum of these and using the relation

$$\psi\frac{\partial^2\psi}{\partial r_i\partial r_j}=\frac{\partial\psi}{\partial r_i}\frac{\partial\psi}{\partial r_j}\quad\text{for}\quad i\neq j. \quad (2.6)$$

In this way the cross terms of the form (2.6) can be eliminated, and T becomes

$$T=\int\bigg\{\frac{\hbar^2(M_\Lambda+M)}{4MM_\Lambda}\left[\left(\frac{\partial\psi}{\partial r_1}\right)^2-\psi\frac{\partial^2\psi}{\partial r_1^2}-\frac{2\psi}{r_1}\frac{\partial\psi}{\partial r_1}+\left(\frac{\partial\psi}{\partial r_2}\right)^2-\psi\frac{\partial^2\psi}{\partial r_2^2}-\frac{2\psi}{r_2}\frac{\partial\psi}{\partial r_2}\right]$$
$$+\frac{\hbar^2}{2M}\left[\left(\frac{\partial\psi}{\partial r_3}\right)^2-\psi\frac{\partial^2\psi}{\partial r_3^2}-\frac{2\psi}{r_3}\frac{\partial\psi}{\partial r_3}\right]\bigg\}r_1r_2r_3dr_1dr_2dr_3. \quad (2.7)$$

With the particular functions (2.4), (2.7) becomes

$$T=\int\bigg\{\frac{\hbar^2(M+M_\Lambda)}{2MM_\Lambda}\left[(\alpha-\beta)^2e^{-(\alpha+\beta)(r_1-D)}-\frac{2}{r_1}f(r_1)f'(r_1)\right]f^2(r_2)g^2(r_3)$$
$$+\frac{\hbar^2}{2M}\left[(\gamma-\delta)^2e^{-(\gamma+\delta)(r_3-D)}-\frac{2}{r_3}g(r_3)g'(r_3)\right]f^2(r_1)f^2(r_2)\bigg\}r_1r_2r_3dr_1dr_2dr_3, \quad (2.8)$$

where a prime on a function denotes derivative with respect to the argument, and use has been made of the symmetry of the wave function ψ in the coordinates r_1 and r_2.

The domain of integration for the variables (r_1,r_2,r_3) must be consistent with the triangular inequalities $r_1+r_2\geqslant r_3$, $r_2+r_3\geqslant r_1$, $r_3+r_1\geqslant r_2$ and the additional restrictions, imposed by the presence of the hard core, that $r_1\geqslant D$, $r_2\geqslant D$, $r_3\geqslant D$. The separation of the domain of integration used here was that proposed in reference 5:

$$\int dr_1dr_2dr_3=\int_D^\infty dr_1\int_D^\infty dr_2\int_D^\infty dr_3-\int_D^\infty dr_1\int_D^\infty dr_2\int_{r_1+r_2}^\infty dr_3-\int_D^\infty dr_2\int_D^\infty dr_3\int_{r_2+r_3}^\infty dr_1-\int_D^\infty dr_3\int_D^\infty dr_1\int_{r_3+r_1}^\infty dr_2. \quad (2.9)$$

[17] For a wave function, such as (2.3), which depends only upon the interparticle separations, all expectation values have a common factor $8\pi^2$ arising from the angle integrals. Since this factor cancels in (2.2), it is omitted from the expressions for the individual expectation values.

With the trial function given in (2.3) and (2.4) and the potentials (2.1), all the expectation values appearing in the variation inequality (2.2) can be expressed in terms of the two basic integrals

$$K(A,B,C)=\int e^{-A(r_1-D)-B(r_2-D)-C(r_3-D)}r_1r_2r_3dr_1dr_2dr_3, \tag{2.10a}$$

$$L(A,B,C)=\int e^{-A(r_1-D)-B(r_2-D)-C(r_3-D)}r_2r_3dr_1dr_2dr_3, \tag{2.10b}$$

which have closed algebraic forms. Expressions for all the expectation values in (2.2) are given in the Appendix in terms of the integrals (2.10).

III. POTENTIAL PARAMETERS

The potential parameters (V_0,η) of the nucleon-nucleon potential (2.1a) were chosen to reproduce the binding energy of the deuteron and the zero-energy triplet scattering length. These potential parameters, which were determined by Ohmura *et al.*,[5] are reproduced in Table I.

The attractive part of a hard-core potential can be characterized by the zero-energy scattering length a^0 and the effective range $r_0{}^0$ which it would have if it were centered at the origin (that is, if $r-D$ were replaced by r). These parameters are related to the scattering length a and the effective range r_0 of the entire potential by[5]

$$a^0=a-D, \tag{3.1a}$$

$$r_0{}^0=(1-D/a)^{-2}(r_0-2D+2D^2/a-2D^3/3a^2). \tag{3.1b}$$

The parameters (a,r_0) of the average Λ-nucleon potential are not known; therefore, Eqs. (3.1) cannot be used to uniquely determine the potential parameters (U_0,λ) in Eq. (2.1b). In this situation, we used the form which Eq. (3.1b) takes in the limiting case $a\rightarrow\infty$:

$$b^0=b-2D, \tag{3.1c}$$

where b is the intrinsic range of the entire potential, and b^0 is that of the attractive well translated to the origin. The value $b=1.5$ F was used; this corresponds to a range of $(\hbar/2M_\pi c)$ for a Yukawa potential without a hard core.[14,18] The values of the range parameter λ in the Λ-nucleon potential (2.1b), which result from this choice of b and the use of (3.1c), are given in Table II.[19]

The range parameters in Tables I and II lead to an appreciably more rapid fall-off of the potentials in the asymptotic region than that to be expected on the basis of the simplest pion-exchange mechanism anticipated in each case. This effect, which becomes more pronounced with larger hard-core radii, is a consequence of the choice of a two-parameter function to represent the attractive well.[20] In order to investigate the possible effects of these compressed (and therefore very deep) attractive wells, a calculation was made (with the single hard-core radius $D=0.4$ F) in which the asymptotic form of the attractive wells for both Λ-nucleon and nucleon-nucleon potentials was taken to correspond to that expected from the relevant pion-exchange mechanisms. This leads to a range parameter $\lambda=2.361$ F^{-1} ($b^0=1.5$ F) for the Λ-nucleon potential and $\eta=1.180$ F^{-1} ($b^0=3.0$ F) for the nucleon-nucleon potential. The depth $V_0=128.9$ MeV was taken for this nucleon-nucleon potential in order to give the correct binding energy of the deuteron.[21] Moreover, a final calculation was made with the Λ-nucleon potential described here and the nucleon-nucleon potential given in Table I for $D=0.4$ F.[22]

IV. RESULTS

The variation expression (2.2) for the depth U_0 of the average Λ-nucleon potential was minimized with respect to the variation parameters $(\alpha,\beta,\gamma,\delta)$ appearing

TABLE I. Nucleon-nucleon potential parameters.

D (F)	V_0 (MeV)	η (F^{-1})
0.2	286.2	1.895
0.4	475.0	2.521
0.6	947.0	3.676

TABLE II. Λ-nucleon potential range parameters.

D (F)	b^0 (F)	λ (F^{-1})
0.2	1.1	3.219
0.4	0.7	5.059
0.6	0.3	11.804

[18] This Λ-nucleon range was chosen as being representative of the lowest-order pion-exchange mechanism which can give rise to a charge-independent Λ-nucleon interaction. Calculations of the Λ-nucleon potentials which arise from simple meson-exchange mechanisms indicate that it is more likely that the observed spin dependence of the Λ-nucleon interaction can be explained in terms of a dominant pion-exchange mechanism than in terms of a dominant kaon-exchange mechanism. See D. B. Lichtenberg and M. Ross, Phys. Rev. **107**, 1714 (1957); **109**, 2163 (1958); and F. Ferrari and L. Fonda, Nuovo Cimento **9**, 842 (1958).

[19] See, for example, J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

[20] For a discussion of more complex hard-core nucleon-nucleon potentials whose asymptotic form is consistent with the one-pion exchange mechanism see, for example, the references in footnote 1.

[21] This nucleon-nucleon potential has a scattering length $a=6.0$ F and an effective range $r_0=2.5$ F; the correct values of these parameters for the triplet nucleon-nucleon potential are $a=5.4$ F and $r_0=1.7$ F.

[22] These potentials are approximately those used in reference 11.

TABLE III. Results of the deuteron variation calculation.

D (F)	γ (F^{-1})	δ (F^{-1})	B_d (MeV)
0.2	0.566	5.58	1.981
0.4	0.600	4.68	1.916
0.6	0.632	5.00	1.665

in (2.4). Three of these parameters were fixed, and U_0 was minimized with respect to the fourth. The varied parameter was subsequently set at its "best" value and U_0 minimized with respect to one of the other parameters. This procedure was carried through the set of four variation parameters and then iterated until the desired accuracy was obtained. The initial values of the nucleon-nucleon parameters (γ,δ) were obtained by using the trial function $g(r)$, given in (2.4b), in a variation calculation to maximize the binding energy of the deuteron. Similarly, the initial values of the Λ-nucleon parameters (α,β) were obtained by minimizing the depth of the Λ-nucleon potential required to give a fictitious bound Λ-nucleon system (the hyperdeuteron) with zero binding energy. The results of these two-body calculations are given in Tables III and IV for the potential parameters given in Tables I and II.

The values $B_d=2.225$ MeV and $B_\Lambda=0.2$ MeV were used in the variation inequality (2.2). The results of the variation calculation are given in Table V for the potential parameters given in Tables I and II. It is interesting to note how close the optimum parameters (β,γ,δ) given in Table V are to those obtained in Tables III and IV for the two-body calculations.[23] That the parameter α differs appreciably in the two-body and three-body calculations is not surprising because this parameter is most sensitive to the separation energy of the Λ particle in each system; and, in the fictitious hyperdeuteron, an incorrect separation energy was assumed.

The results of the variation calculations using the potentials discussed following Table II are given in

TABLE IV. Results of the "hyperdeuteron" variation calculation.

D (F)	α (F^{-1})	β (F^{-1})	U_0 (MeV)
0.2	0.588	7.90	696.5
0.4	0.653	7.49	1677
0.6	0.792	11.78	8835

[23] At first glance, the trend of the optimum parameters δ and β in Tables III, IV, and V as functions of the hard-core radius D appears spurious. The same behavior of the larger of the two nucleon-nucleon parameters was, however, reported in references 5 and 6. Variation calculations of the binding energy of the deuteron with the trial function (2.4b) were made for hard-core radii $D=0.1$ F, 0.3 F, and 0.5 F to supplement the calculations reported in Table III. The optimum parameter γ was found to increase monotonically with increasing hard-core radius; on the other hand, the optimum parameter δ was found to decrease as D increases from 0.1 F to 0.4 F and then to increase as D increases further.

Table VI. In the last two rows of Table VI, the value $\lambda=2.361$ F^{-1} ($b^0=1.5$ F) leads to the same value of the depth U_0 of the Λ-nucleon potential and to nearly equal values for the optimum Λ-nucleon wave function parameters (α,β) for nucleon-nucleon potentials of quite different range and depth. This would seem to indicate that, for a Λ-nucleon potential of given range, the Λ-nucleon parameters are relatively insensitive to the structure of the nucleon part of the hypertriton provided that the nucleon-nucleon potential is consistent with the binding energy of the deuteron.

The Λ-nucleon potential parameters given in Tables V and VI were used to calculate the well-depth parameter s,[19] and the scattering length a and effective range r_0 of the average Λ-nucleon potential[16] according to Eqs. (3.1). These parameters are summarized in Table VII; the last row in this table corresponds to the last two rows in Table VI.

It is inherent in the variation method that the value of U_0 obtained from (2.2) is an overestimate. Previous calculations of U_0 in terms of potentials without hard cores have indicated that reductions in U_0 of the order of 10% might be expected with trial functions of greater flexibility than that given in (2.4).[24] The effect

TABLE V. Results of the variation calculation for the hypertriton.

D (F)	η (F^{-1})	V_0 (MeV)	λ (F^{-1})	U_0 (MeV)	α (F^{-1})	β (F^{-1})	γ (F^{-1})	δ (F^{-1})
0.2	1.895	286.2	3.219	426.0	0.297	7.36	0.547	6.52
0.4	2.521	475.0	5.059	1202	0.325	6.94	0.578	4.55
0.6	3.677	947.0	11.80	7352	0.389	11.28	0.606	4.79

that such an improvement would have on the scattering parameters is indicated in Table VIII, where the values of U_0 were arbitrarily reduced by 10% from the values given in Table VII.

V. CONCLUDING REMARKS

Since the potential $U(r)$ represents the average central, two-body Λ-nucleon interaction effective in the hypertriton when an S state is assumed for each Λ-nucleon pair,[16] it is convenient to characterize $U(r)$ by the S-wave scattering parameters a and r_0. The range of values to be expected for these parameters is indicated in Tables VII and VIII if the variation calculation described here leads to values of U_0 within 10% of the correct ones. An accurate analysis of the hypertriton in terms of Yukawa potentials of intrinsic range $b=1.5$ F without hard cores led to the values[24,25]

$$a\approx -1.5 \text{ F}, \qquad (5.1a)$$

$$r_0\approx \ 2.8 \text{ F}. \qquad (5.1b)$$

[24] B. W. Downs and R. H. Dalitz, Phys. Rev. 114, 593 (1959).
[25] The values (5.1) were taken from results reported in reference 24 for $B_\Lambda=0.25$ MeV, modified slightly to correspond to the current value $B_\Lambda=0.20$ MeV used here. An indication of the

Improvement of the values of U_0 given in the first three rows of Table VII by less than 10% could bring the corresponding scattering lengths into agreement with (5.1a) for any value of the hard-core radius considered here, the corresponding effective ranges (2.4–2.5 F) being somewhat smaller than (5.1b). This suggests the conjecture that (5.1) may provide an approximate characterization of the average Λ-nucleon potential in the hypertriton independent of the value of the hard-core radius for $D \lesssim 0.6$ F. In this connection it should be recalled that the intrinsic ranges b^0 which led to the results in the first three rows of Tables VII and VIII were determined from Eq. (3.1c), which is related to the equation [Eq. (3.1b)] which preserves the effective range of a potential with the introduction of a hard core. Neither the scattering lengths nor the effective ranges listed in the fourth row of Tables VII and VIII bracket the values (5.1). Reduction of the value of U_0 by more than 10% would be required here to bring the scattering length into agreement with (5.1a), and the corresponding value of r_0 would be appreciably larger than (5.1b). It is, of course, possible that the results in the fourth rows of Tables VII and VIII describe the actual situation for $D=0.4$ F better than do those in the second rows. In any case, comparison of the results in the second and fourth rows of these tables indicates the range dependence of the average Λ-nucleon interaction required to reproduce the binding energy of the hypertriton.

TABLE VI. Results of the variation calculation for the hypertriton for potentials having a hard-core radius $D=0.4$ F.

D (F)	η (F^{-1})	V_0 (MeV)	λ (F^{-1})	U_0 (MeV)	α (F^{-1})	β (F^{-1})	γ (F^{-1})	δ (F^{-1})
0.4	2.521	475.0	5.059	1202	0.325	6.94	0.578	4.55
0.4	1.180	128.9	2.361	234.5	0.200	4.40	0.414	2.99
0.4	2.521	475.0	2.361	234.5	0.222	4.30	0.580	4.67

The range parameter λ of the average Λ-nucleon potential, which led to the results in the fourth row of Table VII, is essentially that used by Lichtenberg[11] for a potential of the form (2.1b) with $D=0.4$ F. The value of U_0 that he reported leads to a well-depth parameter $s=0.88$, a scattering length $a=-7.6$ F, and an effective range $r_0=2.7$ F. The values of these parameters indicate that Lichtenberg required a stronger potential than that given by the fourth row of Table VII. A part of this difference is to be expected on account of the error in the work of Ohmura *et al.*,[5,6] upon which Lichtenberg's calculations were based. The rest of the difference is presumably due to the fact that Lichtenberg used a trial function of the form given in (2.3) and (2.4) in which the Λ-nucleon variation parameters (α,β) were taken to be the same as the nucleon-nucleon parameters (γ,δ). The sets of optimum parameters given in the third row of Table VI, which we found in the corresponding case, are not the same.[26]

TABLE VII. Well depth and scattering parameters of Λ-nucleon potentials.

D (F)	U_0 (MeV)	b^0 (F)	s	a (F)	r_0 (F)
0.2	426.0	1.1	0.744	−2.20	2.13
0.4	1202	0.7	0.851	−2.56	2.01
0.6	7352	0.3	0.955	−4.09	1.80
0.4	234.5	1.5	0.762	−3.20	3.32

The results of the present paper can also be compared with those of Dietrich *et al.*,[12] for a hard-core radius $D=0.2$ F. Their results for an attractive square well with an intrinsic range $b^0=1.08$ F lead to a well-depth parameter $s=0.76$, a scattering length $a=-2.6$ F, and an effective range $r_0=1.9$ F for the average Λ-nucleon potential in the hypertriton. These parameters correspond to a slightly stronger potential than that reported in the first row of Table VII, whose attractive well has essentially the same intrinsic range.[27]

For a potential which is characterized by an effective range and a negative scattering length (and, therefore, by a well-depth parameter $s<1$), Eqs. (3.1) imply that the well-depth parameter will be larger, the larger the value of the hard-core radius. This leads one to speculate, as Lichtenberg[11] did, on what value of hard-core radius might be required in order that the bound state of the hypertriton would imply a bound hyperdeuteron. The Λ-nucleon interaction which might lead to a bound hyperdeuteron is the more attractive of the triplet and singlet interactions. If the triplet is the more attractive, then the hyperdeuteron potential is the same as the average potential effective in the hypertriton.[16] The well-depth parameters given in Table VII show that, in this case, the hyperdeuteron would not be expected to exist for a hard-core radius of 0.6 F or less. If the singlet interaction is the more attractive, then the well-depth parameter s_s which would be appropriate to the hyperdeuteron is related to the average well-depth

TABLE VIII. Well depth and scattering parameters of Λ-nucleon potentials of reduced strength.

D (F)	U_0 (MeV)	b^0 (F)	s	a (F)	r_0 (F)
0.2	383.4	1.1	0.670	−1.49	2.47
0.4	1082	0.7	0.766	−1.31	2.58
0.6	6617	0.3	0.860	−0.76	3.49
0.4	211.0	1.5	0.686	−2.08	3.90

shape dependence of the scattering length r_0 is given in reference 14, where results of analyses of the hypertriton in terms of exponential and Yukawa potentials are reported.

[26] It was pointed out by Lichtenberg in reference 11 that an improvement is to be expected if (α,β) are allowed to differ from (γ,δ).

[27] It should be noted that the method of calculation reported in reference 12 is quite different from that used in the present paper; see Sec. I.

parameter s calculated in this paper by[16]

$$s_s = s[3/4 + s_t/4s_s]^{-1}, \tag{5.2}$$

where s_t/s_s is the ratio of the well-depth parameters of the triplet and singlet potentials. Empirical estimates of this ratio, based on analyses of the hypertriton and one other light hypernucleus in terms of Λ-nucleon potentials with an intrinsic range $b=1.5$ F, are 0.45 for potentials without a hard core[24] and 0.55 for potentials with a hard-core radius $D=0.2$ F.[12] Both the value of this ratio and the validity of expression (5.2) depend upon the absence of appreciable three-body Λ-nucleon interactions.[28] Even in the absence of three-body interactions, the ratio s_t/s_s should be determined for each assumed value of the hard-core radius before (5.2) can be applied with any certainty. As an indication of the results which might be obtained, we use (5.2) with the value $s_t/s_s=0.5$ suggested by existing estimates. This leads to $s_s=8s/7$ and to an implied bound hyperdeuteron for $s\geqslant 7/8$. A value of s in excess of this critical value is given in Table VII for $D=0.6$ F; an improvement in this calculation by less than 10% (as indicated in Table VIII) would, however, reduce s below the critical value. These estimates indicate that a hard-core radius of 0.6 F or more would be required in order for the existence of the hypertriton to imply the existence of a bound hyperdeuteron.

An improvement in the results reported here could be expected with the use of a trial function of the form (2.3) with each factor being of the form

$$\{\exp[-\alpha(r-D)]-\exp[-\beta(r-D)]\} + x\{\exp[-\alpha'(r-D)]-\exp[-\beta'(r-D)]\}, \tag{5.3}$$

analogous to the trial function used by Downs and Dalitz[24] for potentials without a hard core. The use of such a trial function would require an appreciably greater computational effort than that expended in the present work, but would not require any modification in the formulation of the variation problem outlined here. The extent of the improvement which can be obtained with a more flexible trial function, such as that suggested above, will have to be known before the conjecture following (5.1) can be taken seriously.

APPENDIX

Explicit expressions for the expectation values appearing in the variation inequality (2.2) are given here in terms of the integrals K and L defined in (2.10).[17] Algebraic expressions for these integrals, which can be obtained from the basic integral

$$I(A,B,C)=\int e^{-A(r_1-D)-B(r_2-D)-C(r_3-D)}dr_1dr_2dr_3$$
$$=\frac{1}{ABC}-\frac{e^{-AD}}{A(A+B)(A+C)}-\frac{e^{-BD}}{B(A+B)(B+C)}-\frac{e^{-CD}}{C(A+C)(B+C)}, \tag{A1}$$

are

$$K(A,B,C)=-e^{(A+B+C)D}\frac{\partial^3}{\partial A\partial B\partial C}e^{-(A+B+C)D}I(A,B,C)=\frac{1}{ABC}\left[D^3+\frac{D^2}{A}+\frac{D^2}{B}+\frac{D^2}{C}+\frac{D}{AB}+\frac{D}{AC}+\frac{D}{BC}+\frac{1}{ABC}\right]$$
$$-\frac{e^{-AD}}{A(A+B)(A+C)}\left[2D^3+\frac{D^2}{A}+\frac{3D^2}{A+B}+\frac{3D^2}{A+C}+\frac{2D}{(A+B)^2}+\frac{2D}{(A+C)^2}+\frac{D}{A(A+C)}+\frac{D}{A(A+B)}+\frac{4D}{(A+C)(A+B)}\right.$$
$$\left.+\frac{2}{(A+B)^2(A+C)}+\frac{2}{(A+B)(A+C)^2}+\frac{1}{A(A+B)(A+C)}\right]-\frac{e^{-BD}}{B(A+B)(B+C)}\left[2D^3+\frac{D^2}{B}+\frac{3D^2}{A+B}+\frac{3D^2}{B+C}\right.$$
$$+\frac{2D}{(A+B)^2}+\frac{2D}{(B+C)^2}+\frac{D}{B(A+B)}+\frac{D}{B(B+C)}+\frac{4D}{(A+B)(B+C)}+\frac{2}{(A+B)^2(B+C)}+\frac{2}{(A+B)(B+C)^2}$$
$$\left.+\frac{1}{B(A+B)(B+C)}\right]-\frac{e^{-CD}}{C(A+C)(B+C)}\left[2D^3+\frac{D^2}{C}+\frac{3D^2}{A+C}+\frac{3D^2}{B+C}+\frac{2D}{(A+C)^2}+\frac{2D}{(B+C)^2}+\frac{D}{C(A+C)}\right.$$
$$\left.+\frac{D}{C(B+C)}+\frac{4D}{(A+C)(B+C)}+\frac{2}{(A+C)^2(B+C)}+\frac{2}{(A+C)(B+C)^2}+\frac{1}{C(A+C)(B+C)}\right] \tag{A2}$$

[28] For a discussion of the effect of three-body Λ-nucleon interactions see, for example, A. R. Bodmer and S. Sampanthar, Nucl. Phys. 31, 251 (1962).

and

$$L(A,B,C)=e^{(A+B+C)D}\frac{\partial^2}{\partial B\partial C}e^{-(A+B+C)D}I(A,B,C)=\frac{1}{ABC}\left[D^2+\frac{D}{B}+\frac{D}{C}+\frac{1}{BC}\right]$$

$$-\frac{e^{-AD}}{A(A+B)(A+C)}\left[D^2+\frac{D}{A+B}+\frac{D}{A+C}+\frac{1}{(A+B)(A+C)}\right]$$

$$-\frac{e^{-BD}}{B(A+B)(B+C)}\left[2D^2+\frac{D}{B}+\frac{D}{A+B}+\frac{3D}{B+C}+\frac{1}{B(B+C)}+\frac{1}{(A+B)(B+C)}+\frac{2}{(B+C)^2}\right]$$

$$-\frac{e^{-CD}}{C(A+C)(B+C)}\left[2D^2+\frac{D}{C}+\frac{D}{A+C}+\frac{3D}{B+C}+\frac{1}{C(B+C)}+\frac{1}{(A+C)(B+C)}+\frac{2}{(B+C)^2}\right]. \quad \text{(A3)}$$

The normalization integral is

$$N=\sum_{i,j,k=1}^{3} d_{ijk}K(A_i,B_j,C_k), \quad \text{(A4)}$$

where

$$A_1=B_1=2\alpha, \quad C_1=2\gamma;$$
$$A_2=B_2=\alpha+\beta, \quad C_2=\gamma+\delta; \quad \text{(A5)}$$
$$A_3=B_3=2\beta, \quad C_3=2\delta.$$

The variation parameters $(\alpha,\beta,\gamma,\delta)$ are defined in (2.4). The expansion coefficients in (A4) are

$$d_{222}=-8, \quad d_{ijk}=4 \text{ if two indices are 2} \quad \text{(A6)}$$

and, otherwise,

$$d_{ijk}=\begin{Bmatrix}1\\-2\end{Bmatrix} \text{ if } i+j+k \text{ is } \begin{Bmatrix}\text{odd}\\\text{even}\end{Bmatrix}.$$

The values of the arguments A_i, B_j, and C_k given in (A5) and those of the expansion coefficients d_{ijk} given in (A6) are used throughout the Appendix.

The expectation value of the kinetic energy is

$$T=\sum_{l=1}^{3} T_l, \quad \text{(A7a)}$$

where

$$T_1=\frac{\hbar^2(M_\Lambda+M)}{2MM_\Lambda}(\alpha-\beta)^2\sum_{j,k=1}^{3} d_{1jk}K(A_2,B_j,C_k)+\frac{\hbar^2}{2M}(\gamma-\delta)^2\sum_{i,j=1}^{3} d_{ij1}K(A_i,B_j,C_2), \quad \text{(A7b)}$$

$$T_2=\frac{\hbar^2(M_\Lambda+M)}{MM_\Lambda}\left\{\alpha\sum_{j,k=1}^{3} d_{1jk}L(A_1,B_j,C_k)+\frac{(\alpha+\beta)}{2}\sum_{j,k=1}^{3} d_{2jk}L(A_2,B_j,C_k)+\beta\sum_{j,k=1}^{3} d_{3jk}L(A_3,B_j,C_k)\right\}, \quad \text{(A7c)}$$

and

$$T_3=\frac{\hbar^2}{M}\left\{\gamma\sum_{i,j=1}^{3} d_{1ij}L(C_1,A_i,B_j)+\frac{(\gamma+\delta)}{2}\sum_{i,j=1}^{3} d_{2ij}L(C_2,A_i,B_j)+\delta\sum_{i,j=1}^{3} d_{3ij}L(C_3,A_i,B_j)\right\}. \quad \text{(A7d)}$$

The expectation value of the nucleon-nucleon potential (2.1a) is

$$V=-V_0\sum_{i,j,k=1}^{3} d_{ijk}K(A_i,B_j,C_k'), \quad \text{(A8a)}$$

where

$$C_k'=C_k+\eta; \tag{A8b}$$

and that of the average Λ-nucleon potential (2.1b) divided by $-U_0$ is

$$P=\sum_{i,j,k=1}^{3} d_{ijk}K(A_i',B_j,C_k), \tag{A9a}$$

where

$$A_i'=A_i+\lambda. \tag{A9b}$$

The parameters η and λ are the potential range parameters given in (2.1).

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Propagation of the Single-Scattering Distribution in Multiple Scattering: Muon Scattering in Iron*

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The moments of the projected angular distribution of the single-scattering process are shown to be derivable from the emergent angular distribution of a beam that has traversed a thick absorber. Since very small deflections do not contribute to the observed moments, ambiguity is avoided by adopting a formulation of the electronic screening that leads to a definite total scattering cross section. The theory is applied to an experiment in which 2-BeV muons are incident on an iron scatterer 18 in. thick. The observed angular distribution is analyzed. It is shown that the nuclear electromagnetic form factor derived from the muon data is consistent with that found from electron scattering, and is completely incompatible with a point-nucleus model.

I. INTRODUCTION

BECAUSE they are thought to interact only with the distribution of charges and currents in an atomic nucleus, charged leptons have been considered excellent probes for a study of the detailed structure of atomic nuclei. Extensive use has already been made of electrons for this purpose.[1] In some respects muons should be even better suited for this task, but until recently the only "beams" of muons available were those of the cosmic rays. A complication also was introduced when muons were reported to scatter[2,3] as predicted by the Molière theory,[4] which is inapplicable if the nucleus cannot be represented by a point charge.

In this paper we describe an experiment designed to study this question. Since it was initiated, however, results have been reported by other investigators that leave little reason to believe that the muon scatters anomalously. Decisive experiments were carried out by Connelly *et al.*,[5] Masek *et al.*,[6] Kim *et al.*,[7] Citron *et al.*,[8] and others. Our results, therefore, are merely confirmatory, but in obtaining them we have introduced a new method for analyzing the data that presumably has utility for many related problems in high-energy physics.

After a beam of particles has penetrated a finite

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